

# A Hamiltonian-Krein (instability) index theory for KdV-like eigenvalue problems

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**Abstract.** The Hamiltonian-Krein (instability) index is concerned with determining the number of eigenvalues with positive real part for the Hamiltonian eigenvalue problem  $\mathcal{J}\mathcal{L}u = \lambda u$ , where  $\mathcal{J}$  is skew-symmetric and  $\mathcal{L}$  is self-adjoint. If  $\mathcal{J}$  has a bounded inverse the index is well-established, and it is given by the number of negative eigenvalues of the operator  $\mathcal{L}$  constrained to act on some finite-codimensional subspace. There is an important class of problems - namely, those of KdV-type - for which  $\mathcal{J}$  does not have a bounded inverse. In this paper we overcome this difficulty and derive the index for eigenvalue problems of KdV-type. We use the index to discuss the spectral stability of homoclinic traveling waves for KdV-like problems and BBM-type problems.

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## 1. INTRODUCTION

We consider the spectral problem of the form

$$\partial_x \mathcal{L}u = \lambda u, \quad (1.1)$$

where  $\mathcal{L}$  is a self-adjoint linear differential Fredholm operator with zero index and with domain  $D(\mathcal{L}) = H^s(\mathbb{R})$  for some  $s \geq 0$ . Eigenvalue problems of this type readily arise, e.g., when considering the stability of waves to KdV-like problems. We will furthermore assume that  $\mathcal{L} = \mathcal{L}_0 + \mathcal{K}$ , where  $\mathcal{K}$  is relatively compact perturbation of  $\mathcal{L}_0$ , and  $\mathcal{L}_0$  is a “constant coefficient” strongly elliptic operator<sup>1</sup> given by  $\widehat{\mathcal{L}_0 f}(\xi) = q(\xi)\hat{f}(\xi)$ . It will be assumed that for the operator  $\mathcal{L}$ ,

- (a) there are  $n(\mathcal{L}) < +\infty$  negative eigenvalues (counting multiplicity), and each of the corresponding eigenvectors  $\{f_j\}_{j=1}^{n(\mathcal{L})}$  belong to  $H^{1/2}(\mathbb{R})$ .
- (b)  $\sigma_{\text{ess}}(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}_0) = \text{Range}(q) \subset [\kappa^2, +\infty)$ ,  $\kappa > 0$
- (c)  $\dim[\ker(\mathcal{L})] = 1$  with  $\ker(\mathcal{L}) = \text{span}\{\psi_0\}$ , and  $\psi_0$  is real-valued and  $\psi_0 \in H^\infty(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ .

For the precise definitions of the various Sobolev spaces, consult [Section 2](#).

The goal of this paper is to compute an instability index, hereafter known as the Hamiltonian-Krein index, for the eigenvalue problem (1.1). In the derivation of instability indices for eigenvalue problems of the form

$$\mathcal{J}\mathcal{L}u = \lambda u,$$

where  $\mathcal{J}$  is skew-symmetric and  $\mathcal{L}$  is symmetric, it was crucial in previous works that  $\mathcal{J}$  have a bounded inverse on (at minimum) a finite co-dimensional space (e.g., see [\[6, 8, 11, 12\]](#)). Define the standard inner-product on  $L^2(\mathbb{R})$  by

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)\overline{g}(x)dx.$$

It is clear that the operator  $\partial_x$  is skew-symmetric on  $L^2(\mathbb{R})$ ; however, it does not have a bounded inverse. This is a reflection of the fact that  $\sigma(\partial_x) = \sigma_{\text{ess}}(\partial_x) = i\mathbb{R}$ . The aim of this paper is to overcome this obstacle. Briefly, this will be accomplished by reducing the eigenvalue problem (1.1) to an equivalent problem for which the operator  $\mathcal{J}$  does have a bounded inverse. However, by doing so it will be the case that for the new operator  $\mathcal{L}$ :

- (a) the essential spectrum will (generically) be  $[0, \infty)$ , which violates the assumption present in the original computation of the Hamiltonian-Krein index that the essential spectrum be bounded away from the origin
- (b) the negative index of the new  $\mathcal{L}$ , which is needed in the evaluation of the index, is not obvious.

Both of these obstacles must be overcome before coming to the final conclusion of [Theorem 4.3](#).

The paper is organized in the following manner. In [Section 2](#) we discuss some preliminary ideas which will be needed in the analysis. The results presented therein are not new, and are included solely to help make the paper more accessible. In [Section 3](#) the equivalent eigenvalue problem is derived, and properties of the new operator  $\mathcal{L}$  are given. [Section 4](#) contains the main result of the paper. In [Section 5](#) we give a couple of applications of the theoretical result, and compare the results here with what is already known in the literature.

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<sup>1</sup>that is,  $q(\xi) \geq \kappa^2$  for some  $\kappa > 0$

## 2. PRELIMINARIES

Define the Fourier transform and its inverse via the formulas

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx, \quad f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi,$$

which are valid for functions in the Schwartz class  $\mathcal{S}$ . Introduce fractional order differential operators via the Fourier transform, i.e. for  $s \geq 0$ ,

$$|\widehat{\partial_x^s f}(\xi)| := (2\pi)^s |\xi|^s \hat{f}(\xi).$$

The norm of the Sobolev space  $H^s(\mathbb{R})$ ,  $s \geq 0$ , is given by

$$\|f\|_{H^s} := \left( \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + \xi^2)^s d\xi \right)^{1/2}.$$

The space of infinitely smooth functions (with  $L^2(\mathbb{R})$  decay of all derivatives),  $H^\infty := \cap_{s=1}^\infty H^s$  is not a Banach space, but it has a well-understood Frechet space structure.

We also need to consider operators in the form  $|\partial_x|^{-\alpha}$  for some  $\alpha > 0$ . Regarding Sobolev spaces of negative order, we introduce the norm

$$\|f\|_{\dot{H}^{-\alpha}} := \left( \int_{\mathbb{R}} \frac{|\hat{f}(\xi)|^2}{|\xi|^{2\alpha}} d\xi \right)^{1/2},$$

and say that a Schwartz function  $f$  belongs to  $\dot{H}^{-\alpha}(\mathbb{R})$  if  $\|f\|_{\dot{H}^{-\alpha}}$  is finite. The Banach space  $\dot{H}^{-\alpha}(\mathbb{R})$  is obtained as the completion of the Schwartz class  $\mathcal{S}$  in this norm. Note that the Sobolev spaces of negative order will in general contain distributions<sup>2</sup>. Further note that

(a)  $|\partial_x|^{-\alpha} : \dot{H}^{-\alpha}(\mathbb{R}) \mapsto L^2(\mathbb{R})$  is an isometry

(b)  $|\partial_x|^\alpha : L^2(\mathbb{R}) \mapsto \dot{H}^{-\alpha}(\mathbb{R})$  is an isometry.

Some of these operators have a nice representation as fractional integrals. For example, (again for Schwartz functions)

$$|\partial_x|^{-1/2} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1/2}} dy;$$

in particular,  $|\partial_x|^{-1/2} f$  is real-valued if  $f$  is. Note that unless  $f$  has extra cancellation properties<sup>3</sup>, then for large values of  $x$  one has  $|\partial_x|^{-1/2} f \sim |x|^{-1/2}$  and hence  $|\partial_x|^{-1/2} f \notin L^2(\mathbb{R})$ .

Finally, note that for  $f \in \dot{H}^{-1}(\mathbb{R})$  we may define the operator  $\partial_x^{-1}$  via

$$\widehat{\partial_x^{-1} f}(\xi) = -\frac{1}{2\pi i \xi} \hat{f}(\xi).$$

The operator  $\partial_x^{-1}$  is skew-symmetric, as may be seen by the Plancherel's theorem. In particular, for every real-valued  $f \in \dot{H}^{-1}(\mathbb{R}) \cap L^2(\mathbb{R})$  we have that  $\langle \partial_x^{-1} f, f \rangle = 0$ . One may identify  $\dot{H}^{-1}(\mathbb{R})$  as the space of distributional derivatives  $\partial_x(L^2(\mathbb{R})) \subset \mathcal{S}'$ . More precisely,

$$\dot{H}^{-1}(\mathbb{R}) = \partial_x(L^2(\mathbb{R})) = \{h : h = \partial_x f \in \mathcal{S}', f \in L^2(\mathbb{R})\}, \quad \|h\|_{\dot{H}^{-1}} := \|f\|_{L^2}.$$

<sup>2</sup>In fact, one may define  $\dot{H}^{-\alpha}$  as the dual space to  $\dot{H}^\alpha$ , with the obvious definitions. In doing that, one needs to be careful since the “norm”  $\|\cdot\|_{\dot{H}^\alpha}$  assigns zero value to the constant functions and thus, those functions need to be mod-ed out.

<sup>3</sup>At a minimum  $\int f = 0$ , but actually more, like  $f \in \mathcal{H}^1(\mathbb{R})$  - the Hardy space on the line

## 2.1. Littlewood-Paley operators

Let  $\zeta \in C_0^\infty(\mathbb{R})$  be a positive and even cut-off function which satisfies

$$\zeta(z) = \begin{cases} 1, & |z| < 1 \\ 0, & |z| > 2. \end{cases}$$

For  $a > 0$  define the Littlewood-Paley operator  $P_{<a}$  via

$$\widehat{P_{<a}f}(\xi) = \zeta(\xi/a)\hat{f}(\xi).$$

Naturally, we take  $P_{\geq a} = \mathcal{I} - P_{<a}$ , where  $\mathcal{I}$  is the identity operator. The related operators  $P_{\sim a}$  are defined via  $P_{\sim a} := P_{<a} - P_{<\frac{a}{2}}$ . Alternatively, let  $\varphi(z) := \zeta(z) - \zeta(2z)$  and let  $\widehat{P_{\sim a}f}(\xi) = \varphi(\xi/a)\hat{f}(\xi)$ . Note that by the Hardy-Littlewood-Sobolev inequality, we have for all  $1 \leq p \leq \infty$

$$\|P_{<a}f\|_{L^p} + \|P_{\geq a}f\|_{L^p} + \|P_{\sim a}f\|_{L^p} \leq C(1 + \|\hat{\zeta}\|_{L^1})\|f\|_{L^p}.$$

We will often denote<sup>4</sup>

$$f_{<a} := P_{<a}f, \quad f_{\geq a} := P_{\geq a}f, \quad f_{\sim a} = P_{\sim a}f.$$

Note that the operators  $P_{\sim a}$  provide a useful partition of unity. Indeed, note that  $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1$  for  $\xi \neq 0$ , and as a consequence

$$\mathcal{I} = \sum_{k=-\infty}^{\infty} P_{\sim 2^k} = P_{<1} + \sum_{k=1}^{\infty} P_{\sim 2^k}.$$

A version of the Sobolev embedding estimates (also known as Bernstein inequalities) is given by

$$\|P_{\sim 2^k}f\|_{L^q} \leq C2^{k(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}. \quad (2.1)$$

for all  $1 \leq p < q \leq \infty$ .

We have the following lemma:

**Lemma 2.1.** *The subspace  $\{|\partial_x|^{1/2}g : g \in H^{1/2}(\mathbb{R})\}$  is dense in  $L^2(\mathbb{R})$ .*

**Proof:** Let  $\varepsilon > 0$  and  $f \in L^2(\mathbb{R})$  be given function. Then there exists  $\delta > 0$  so that

$$\int_{-\delta}^{\delta} |\hat{f}(\xi)|^2 d\xi \leq \varepsilon^2.$$

Define

$$\hat{g}(\xi) := \begin{cases} c\varepsilon, & |\xi| \leq \delta \\ \hat{f}(\xi)/\sqrt{2\pi|\xi|}, & |\xi| > \delta. \end{cases}$$

It follows that  $g \in H^{1/2}(\mathbb{R})$  - in fact,  $\| |\partial_x|^{1/2}g \|_{L^2}^2 = \int_{|\xi|>\delta} |\hat{f}(\xi)|^2 d\xi \leq \|f\|_{L^2}^2$  - while  $\|g\|_{L^2}^2 \leq \|f\|_{L^2}^2/(2\pi\delta)$ . In addition, by Plancherel's

$$\|f - |\partial_x|^{1/2}g\|_{L^2}^2 = \int_{-\delta}^{\delta} |\hat{f}(\xi)|^2 d\xi \leq \varepsilon^2. \quad \square$$

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<sup>4</sup>By slight abuse of notations, we will always use  $\widehat{f_{\sim a}}(\xi) := \Phi(\xi/a)\hat{f}(\xi)$  and  $\Phi$  is supported around 1 smooth function

### 3. THE EQUIVALENT EIGENVALUE PROBLEM

#### 3.1. The reformulation

We proceed with the reformulation of the eigenvalue problem (1.1). We first note that for nonzero eigenvalues it will necessarily be the case that  $u \in \dot{H}^{-1}(\mathbb{R})$  if  $u \in D(\mathcal{L}) = H^s(\mathbb{R})$ . Indeed, from (1.1)

$$\|u\|_{\dot{H}^{-1}} = \frac{1}{|\lambda|} \|\mathcal{L}u\|_{L^2}.$$

This observation motivates the following change of variables. Set

$$u = |\partial_x|^{1/2} v \quad \Leftrightarrow \quad v = |\partial_x|^{-1/2} u.$$

Note that  $v \in \dot{H}^{-1/2}(\mathbb{R}) \cap H^{s+1/2}(\mathbb{R})$ , since  $|\partial_x|^{-1/2} : \dot{H}^{-1}(\mathbb{R}) \cap H^s(\mathbb{R}) \mapsto \dot{H}^{-1/2}(\mathbb{R}) \cap H^{s+1/2}(\mathbb{R})$  is a bounded map. The eigenvalue problem for  $v$  becomes

$$\partial_x |\partial_x|^{-1/2} \mathcal{L} |\partial_x|^{1/2} v = \lambda v,$$

which can be massaged to

$$\partial_x |\partial_x|^{-1} \cdot |\partial_x|^{1/2} \mathcal{L} |\partial_x|^{1/2} v = \lambda v.$$

Upon introducing the new operators

$$\mathcal{J} := \partial_x |\partial_x|^{-1}, \quad \mathcal{L}^\diamond := |\partial_x|^{1/2} \mathcal{L} |\partial_x|^{1/2}, \quad (3.1)$$

we now see that (1.1) for  $u \in L^2(\mathbb{R})$  can be rewritten as

$$\mathcal{J} \mathcal{L}^\diamond v = \lambda v, \quad v \in \dot{H}^{-1/2}(\mathbb{R}) \cap H^{s+1/2}(\mathbb{R}). \quad (3.2)$$

Consider the operator  $\mathcal{L}^\diamond$ . Clearly, while (3.1) specifies the action of  $\mathcal{L}^\diamond$  on smooth vectors, it does not address the important issue of whether or not  $\mathcal{L}^\diamond$  is self-adjoint<sup>5</sup>. For this, one needs to specify a domain. We would like to point out that there are several (potentially different ways) to obtain a self-adjoint extension. For the purposes of this section, we proceed in a canonical way, by building the Friedrich's extension. We will however give a more direct construction in Section 5. We follow the arguments in [18, Theorem VIII.15]. More concretely, consider the bilinear form

$$q(f, g) := \langle |\partial_x|^{1/2} \mathcal{L} (|\partial_x|^{1/2} f), g \rangle = \langle \mathcal{L} |\partial_x|^{1/2} f, |\partial_x|^{1/2} g \rangle.$$

According to [18, Theorem VIII.15], if we show that the quadratic form  $q$  is semi-bounded (that is  $q(f, f) \geq -M\|f\|^2$  for some  $M$ ), then  $q$  is the quadratic form of a unique self-adjoint operator, the Friedrich's extension, which we call again  $\mathcal{L}^\diamond$ , with domain  $D(\mathcal{L}^\diamond) = \{f \in H^{s+1}(\mathbb{R}) : \mathcal{L}^\diamond f \in L^2(\mathbb{R})\} \subset H^{s+1}(\mathbb{R})$ . Let  $\{f_j\}_{j=1}^N$  be a normalized basis of the finite-dimensional negative subspace of  $\mathcal{L}$ , i.e.  $\mathcal{L} f_j = -\mu_j^2 f_j, j = 1, \dots, N$ . In order to show the semi-boundedness of  $q$ , decompose

$$|\partial_x|^{1/2} f = h + \sum_{j=1}^N \langle |\partial_x|^{1/2} f, f_j \rangle f_j = h + \sum_{j=1}^N \langle f, |\partial_x|^{1/2} f_j \rangle f_j,$$

where  $\langle \mathcal{L} h, h \rangle \geq 0$ , since  $h \in \text{span}[f_1, \dots, f_N]^\perp$ . We have that

$$q(f, f) = \langle \mathcal{L} |\partial_x|^{1/2} f, |\partial_x|^{1/2} f \rangle = \langle \mathcal{L} h, h \rangle - \sum_{j=1}^N \mu_j^2 \langle f, |\partial_x|^{1/2} f_j \rangle^2 \geq -M\|f\|^2,$$

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<sup>5</sup>even though it is clearly a symmetric operator

where  $M = N \sup_{j \in [1, N]} (\mu_j^2 \|f_j\|_{H^{1/2}}^2)$ . Thus, we have constructed the Friedrich's extension of  $\mathcal{L}^\diamond$  by virtue of [18, Theorem VIII.15].

Next, several comments are in order regarding the operator  $\mathcal{J}$ . Not only is this operator skew-symmetric on  $L^p(\mathbb{R})$  for any  $1 < p < +\infty$ , it is a classical operator, well-studied in the literature; namely, the Hilbert transform. It can be alternatively defined (on Schwartz functions) via the formula

$$\widehat{\mathcal{J}f}(\xi) = -i \operatorname{sign}(\xi) \hat{f}(\xi),$$

or it can be defined as the singular integral

$$\mathcal{J}f(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

Unlike the operator  $\partial_x$ , the Hilbert transform is a bounded operator on a variety of function spaces; in particular, on all  $L^p(\mathbb{R})$  for  $1 < p < +\infty$ . Furthermore, on  $L^2(\mathbb{R})$  it is the case that  $\mathcal{J} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$  is an isometry with  $(\mathcal{J})^{-1} = -\mathcal{J}$  [13, Chapter 16.3.2].

We will concentrate our interest on the spectral stability/instability of the linear system (1.1). We say that the linearized problem (1.1) is (spectrally) unstable if there is a  $\lambda$  with positive real part and a corresponding function  $u \in D(L) \cap H^\infty(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$  so that (1.1) is satisfied in classical sense. Otherwise, the problem is spectrally stable. Clearly, spectral instability/stability is equivalent to the existence (non-existence, respectively) of solutions  $v$  to (3.2) with  $\operatorname{Re} \lambda > 0$ . In conclusion, the eigenvalue problem (3.2) is the correct one to study in order to apply the previous Hamiltonian-Krein (instability) index theorems. The application of these theories will require a careful study of the operator  $\mathcal{L}^\diamond$ .

### 3.2. Relation between the point spectrums of $\mathcal{L}$ and $\mathcal{L}^\diamond$

Before we relate the negative spectrum of the sandwiched operator  $\mathcal{L}^\diamond$  to that of  $\mathcal{L}$ , we must first understand the kernel of  $\mathcal{L}^\diamond$ .

**Lemma 3.1.** *Regarding the operator  $\mathcal{L}^\diamond$  we have that  $\dim[\ker(\mathcal{L}^\diamond)] = 1$  with  $\ker(\mathcal{L}^\diamond) = \operatorname{span}\{|\partial_x|^{-1/2}\psi_0\}$ .*

**Proof:** Since  $\psi_0 \in \dot{H}^{-1}(\mathbb{R}) \cap H^\infty(\mathbb{R})$ , it is the case that  $|\partial_x|^{-1/2}\psi_0 \in \dot{H}^{-1/2}(\mathbb{R}) \cap H^\infty(\mathbb{R})$ . Since

$$\mathcal{L}|\partial_x|^{1/2}(|\partial_x|^{-1/2}\psi_0) = \mathcal{L}\psi_0 = 0,$$

it is then clear that  $\dim[\ker(\mathcal{L}^\diamond)] \geq 1$ . In order to determine if the kernel is any larger, consider  $\mathcal{L}^\diamond u = 0$  as an equality of  $L^2(\mathbb{R})$  functions. Testing this equation against all functions  $v \in H^{1/2}(\mathbb{R})$  yields

$$\langle \mathcal{L}^\diamond u, v \rangle = \langle |\partial_x|^{1/2} \mathcal{L}|\partial_x|^{1/2} u, v \rangle = \langle \mathcal{L}|\partial_x|^{1/2} u, |\partial_x|^{1/2} v \rangle = 0.$$

Because of the density Lemma 2.1 we can rewrite the above as

$$\langle \mathcal{L}|\partial_x|^{1/2} u, w \rangle = 0, \quad w \in L^2(\mathbb{R}).$$

Consequently, it must be the case that (as an equality of  $L^2(\mathbb{R})$  functions)

$$\mathcal{L}|\partial_x|^{1/2} u = 0 \quad \Rightarrow \quad u = C|\partial_x|^{-1/2}\psi_0.$$

The desired conclusion has now been achieved.  $\square$

Now that we see the kernel of the sandwiched operator is no larger than the kernel of the original operator, the next thing to be understood is the generalized kernel of  $\mathcal{J}\mathcal{L}^\diamond$ ,  $\operatorname{gker}(\mathcal{J}\mathcal{L}^\diamond)$ .

**Lemma 3.2.** *Suppose that  $\psi_0 \in \dot{H}^{-1}(\mathbb{R})$  satisfies*

$$\langle \mathcal{L}^{-1}(\partial_x^{-1}\psi_0), \partial_x^{-1}\psi_0 \rangle \neq 0.$$

*The generalized kernel is then given by*

$$\operatorname{gker}(\mathcal{J}\mathcal{L}^\diamond) = \operatorname{span}\{|\partial_x|^{-1/2}\psi_0, |\partial_x|^{-1/2}\mathcal{L}^{-1}\partial_x^{-1}\psi_0\}.$$

*Remark 3.3.* Since  $\mathcal{L}$  has a nontrivial kernel, it is not clear that the expression  $\mathcal{L}^{-1}(\partial_x^{-1}\psi_0)$  is valid. Since  $\partial_x^{-1}$  is a skew-symmetric operator, it is the case that  $\langle \partial_x^{-1}\psi_0, \psi_0 \rangle = 0$ . The fact that  $\mathcal{L}$  is self-adjoint, and the additional fact that  $\partial_x^{-1} \in \ker(\mathcal{L})^\perp$ , then tells us that the expression makes sense.

**Proof:** Since  $\mathcal{J}$  has bounded inverse, we know from Lemma 3.1 that  $\ker(\mathcal{J}\mathcal{L}^\diamond) = \text{span}\{|\partial_x|^{-1/2}\psi_0\}$ . The first element in the generalized kernel is then found by solving

$$\mathcal{J}\mathcal{L}^\diamond u = |\partial_x|^{-1/2}\psi_0 \quad \Rightarrow \quad \mathcal{L}^\diamond u = |\partial_x|^{1/2}\partial_x^{-1}\psi_0.$$

Since  $\psi_0 \in \dot{H}^{-1}(\mathbb{R})$ , the expression on the right makes sense. We would like to begin to use the Fredholm solvability theory at this point, but unfortunately the fact that the origin is not necessarily isolated from the (essential) spectrum of  $\mathcal{L}^\diamond$  means that this is not possible. However, the form of  $\mathcal{L}^\diamond$  means that the above is equivalent to

$$|\partial_x|^{1/2}\mathcal{L}|\partial_x|^{1/2}u = |\partial_x|^{1/2}\partial_x^{-1}\psi_0 \quad \Rightarrow \quad \mathcal{L}|\partial_x|^{1/2}u = \partial_x^{-1}\psi_0.$$

The equality on the right follows from the fact that if  $|\partial_x|^{1/2}G = 0$  for an  $L^2$  function  $G$  (in the sense of distributions), then  $G = 0$ . Since  $\partial_x^{-1}\psi_0 \in \ker(\mathcal{L})^\perp$ , by the Fredholm solvability theory the above has a solution. The second element in the Jordan chain is given by

$$u = |\partial_x|^{-1/2}\mathcal{L}^{-1}\partial_x^{-1}\psi_0.$$

The result is proven once it is shown that the Jordan chain is no longer. Upon continuing we see that the next element in the Jordan chain, if it exists, is found by solving

$$\mathcal{J}\mathcal{L}^\diamond u = |\partial_x|^{-1/2}\mathcal{L}^{-1}\partial_x^{-1}\psi_0 \quad \Rightarrow \quad \mathcal{L}|\partial_x|^{1/2}u = \partial_x^{-1}\mathcal{L}^{-1}\partial_x^{-1}\psi_0.$$

The Fredholm solvability theory requires that

$$0 = \langle \partial_x^{-1}\mathcal{L}^{-1}\partial_x^{-1}\psi_0, \psi_0 \rangle = -\langle \mathcal{L}^{-1}\partial_x^{-1}\psi_0, \partial_x^{-1}\psi_0 \rangle.$$

By assumption this equality cannot hold, which completes the proof.  $\square$

Now that the structure of the kernel is well-understood (a crucial ingredient in the index theories), we now turn to the problem of the negative index for the operator  $\mathcal{L}^\diamond$ , say  $n(\mathcal{L}^\diamond)$ . In general, we let  $n(\mathcal{S})$  denote the number of negative eigenvalues (counting multiplicity) of the self-adjoint operator  $\mathcal{S}$ . Recalling the assumption (b) on the essential spectrum of  $\mathcal{L}$ , we have the following:

**Lemma 3.4.** Assume that  $\sigma_{\text{ess}}(\mathcal{L}^\diamond) \subset [0, +\infty)$ . If  $n(\mathcal{L}) < +\infty$ , then  $n(\mathcal{L}) = n(\mathcal{L}^\diamond)$ .

**Proof:** Let  $n(\mathcal{L}) = N$ , let  $-\lambda_N^2 \leq -\lambda_{N-1}^2 \leq \dots \leq -\lambda_1^2 < 0$  denote the negative eigenvalues, and let  $f_1, \dots, f_N$  denote the corresponding eigenfunctions. As a consequence of the Courant max/min principle it is known that

$$f^\perp \in \text{span}\{f_1, \dots, f_N\}^\perp \quad \Rightarrow \quad \langle \mathcal{L}f^\perp, f^\perp \rangle \geq 0.$$

Set  $g_j = |\partial_x|^{1/2}f_j$ , and let  $g^\perp \in \text{span}\{g_1, \dots, g_N\}^\perp$  be given. For each  $j = 1, \dots, N$  we have

$$\langle |\partial_x|^{1/2}g^\perp, f_j \rangle = \langle g^\perp, g_j \rangle = 0,$$

so that  $|\partial_x|^{1/2}g^\perp \in \text{span}\{f_1, \dots, f_N\}^\perp$ . Consequently, we have that

$$\langle \mathcal{L}|\partial_x|^{1/2}g^\perp, |\partial_x|^{1/2}g^\perp \rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{L}^\diamond g^\perp, g^\perp \rangle \geq 0.$$

In other words, the negative subspace of  $\mathcal{L}^\diamond$ , i.e., the subspace of  $\mathcal{L}^\diamond$  which corresponds to the negative eigenvalues of  $\mathcal{L}^\diamond$ , must be a subspace of the negative subspace of  $\mathcal{L}$ . In conclusion, we have that  $n(\mathcal{L}^\diamond) \leq N$ .

Now that it is known that  $n(\mathcal{L}^\diamond)$  is finite, assume that  $n(\mathcal{L}^\diamond) = M$ . Equality of the two indices for  $M = 0$  follows immediately from Lemma 2.1, so assume  $M \geq 1$ . We first show that all eigenfunctions of  $\mathcal{L}^\diamond$  corresponding to non-zero eigenvalues belong to  $\dot{H}^{-1/2}(\mathbb{R})$ . Indeed, let  $\mu \neq 0$  be an eigenvalue, with eigenfunction  $f$ , so that

$$\mu f = \mathcal{L}^\diamond f = |\partial_x|^{1/2}(\mathcal{L}|\partial_x|^{1/2}f).$$

Since  $|\partial_x|^{1/2} : L^2(\mathbb{R}) \mapsto \dot{H}^{-1/2}(\mathbb{R})$  is an isometry, the result now follows.

Next, let  $f_1, \dots, f_M$  be the normalized eigenfunctions of  $\mathcal{L}^\diamond$  which correspond to the negative eigenvalues  $-\mu_M^2 \leq -\mu_{N-1}^2 \leq \dots \leq -\mu_1^2 < 0$ . For  $j = 1, \dots, M$  set  $g_j = |\partial_x|^{-1/2} f_j \in L^2(\mathbb{R})$  (in fact  $\|g_j\|_{L^2} = \|f_j\|_{\dot{H}^{-1/2}}$ ) and fix  $g \in \text{span}\{g_1, \dots, g_M\}^\perp$  so that  $\|g\|_{H^s} \leq 1$ . For  $0 < \varepsilon < 1$ , we have

$$\langle \mathcal{L}g, g \rangle = \langle \mathcal{L}g_{>\varepsilon}, g_{>\varepsilon} \rangle + 2\langle \mathcal{L}g_{>\varepsilon}, g_{\leq\varepsilon} \rangle + \langle \mathcal{L}g_{\leq\varepsilon}, g_{\leq\varepsilon} \rangle.$$

By using Cauchy-Schwartz the latter two terms can be bounded via

$$2|\langle \mathcal{L}g_{>\varepsilon}, g_{\leq\varepsilon} \rangle| + |\langle \mathcal{L}g_{\leq\varepsilon}, g_{\leq\varepsilon} \rangle| \leq C\|g\|_{H^s}\|g_{\leq\varepsilon}\|_{L^2} = C\|g_{\leq\varepsilon}\|_{L^2},$$

where we have used that<sup>6</sup>  $\|\mathcal{L}g\|_{L^2} \leq C\|g\|_{H^s}$ . Regarding the first term  $\langle \mathcal{L}g_{>\varepsilon}, g_{>\varepsilon} \rangle$ , write

$$\langle \mathcal{L}g_{>\varepsilon}, g_{>\varepsilon} \rangle = \langle \mathcal{L}|\partial_x|^{1/2}|\partial_x|^{-1/2}g_{>\varepsilon}, |\partial_x|^{1/2}|\partial_x|^{-1/2}g_{>\varepsilon} \rangle = \langle \mathcal{L}^\diamond|\partial_x|^{-1/2}g_{>\varepsilon}, |\partial_x|^{-1/2}g_{>\varepsilon} \rangle$$

Projecting  $|\partial_x|^{-1/2}g_{>\varepsilon}$  over the spectrum of  $\mathcal{L}^\diamond$  yields

$$|\partial_x|^{-1/2}g_{>\varepsilon} = h_\varepsilon + \sum_{j=1}^M \langle |\partial_x|^{-1/2}g_{>\varepsilon}, f_j \rangle f_j,$$

where  $\langle \mathcal{L}^\diamond h_\varepsilon, h_\varepsilon \rangle \geq 0$  and  $h_\varepsilon \in \text{span}\{f_1, \dots, f_M\}^\perp$ . Since

$$0 = \langle g, g_j \rangle = \langle |\partial_x|^{-1/2}f_j, g_{>\varepsilon} \rangle + \langle g_j, g_{\leq\varepsilon} \rangle = \langle f_j, |\partial_x|^{-1/2}g_{>\varepsilon} \rangle + \langle g_j, g_{\leq\varepsilon} \rangle,$$

we can rewrite the above expansion as

$$|\partial_x|^{-1/2}g_{>\varepsilon} = h_\varepsilon - \sum_{j=1}^M \langle g_j, g_{\leq\varepsilon} \rangle f_j.$$

It then follows that

$$\langle \mathcal{L}^\diamond|\partial_x|^{-1/2}g_{>\varepsilon}, |\partial_x|^{-1/2}g_{>\varepsilon} \rangle = \langle \mathcal{L}^\diamond h_\varepsilon, h_\varepsilon \rangle - \sum_{j=1}^M \mu_j^2 |\langle g_j, g_{\leq\varepsilon} \rangle|^2.$$

Using the definition of  $\mathcal{L}^\diamond$  we can rewrite the above as

$$\langle \mathcal{L}g_{>\varepsilon}, g_{>\varepsilon} \rangle = \langle \mathcal{L}^\diamond h_\varepsilon, h_\varepsilon \rangle - \sum_{j=1}^M \mu_j^2 |\langle g_j, g_{\leq\varepsilon} \rangle|^2.$$

Again using Cauchy-Schwartz we have that  $|\langle g_j, g_{\leq\varepsilon} \rangle| \leq \|g_j\|_{L^2}\|g_{\leq\varepsilon}\|_{L^2} \leq C\|g_{\leq\varepsilon}\|_{L^2}$ , where  $C = \sup_{j \in [1, N]} \|f_j\|_{\dot{H}^{-1/2}}$ . It follows that

$$\langle \mathcal{L}g_{>\varepsilon}, g_{>\varepsilon} \rangle \geq \langle \mathcal{L}^\diamond h_\varepsilon, h_\varepsilon \rangle - C\|g_{\leq\varepsilon}\|_{L^2}^2.$$

In addition, note that by Cauchy-Schwartz

$$\langle \mathcal{L}g_{\leq\varepsilon}, g_{\leq\varepsilon} \rangle \leq \|\mathcal{L}g_{\leq\varepsilon}\|_{L^2}\|g_{\leq\varepsilon}\|_{L^2} \leq C\|g_{\leq\varepsilon}\|_{H^s}\|g_{\leq\varepsilon}\|_{L^2} \leq C\|g_{\leq\varepsilon}\|_{L^2}^2.$$

Putting everything together yields

$$\langle \mathcal{L}g, g \rangle \geq -C\|g_{\leq\varepsilon}\|_{L^2}(1 + \|g_{\leq\varepsilon}\|_{L^2}).$$

Since  $\varepsilon > 0$  is arbitrary and  $\lim_{\varepsilon \rightarrow 0} \|g_{\leq\varepsilon}\|_{L^2} = 0$ , it must then be the case that  $\langle \mathcal{L}g, g \rangle \geq 0$ . This inequality implies that  $n(\mathcal{L}) \leq M = n(\mathcal{L}^\diamond)$ . The proof is now complete.  $\square$

<sup>6</sup>Note that we assume that  $\mathcal{L} : D(\mathcal{L}) \subset H^s \rightarrow L^2$  and hence the estimate  $\|\mathcal{L}g\|_{L^2} \leq C\|g\|_{H^s}$



### 3.3. An example

In the previous section, we have considered the theoretical relation between the spectral properties of  $\mathcal{L}$  and  $\mathcal{L}^\diamond$ . We would like now to explore it further for a specific example.

**Proposition 3.5.** *Let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be  $C^1$  smooth and (sufficiently) decaying potential. Consider the corresponding Schrödinger operator  $\mathcal{L} := -\partial_x^2 + c - V$ , with  $c > 0$  and  $D(\mathcal{L}) = H^2(\mathbb{R})$ . The sandwiched operator  $\mathcal{L}^\diamond := |\partial_x|^{1/2} \mathcal{L} |\partial_x|^{1/2}$  with domain  $D(\mathcal{L}^\diamond) = H^3(\mathbb{R})$  is self-adjoint, and moreover  $\sigma_{\text{ess}}(\mathcal{L}^\diamond) = [0, \infty)$ , while  $n(\mathcal{L}^\diamond) = n(\mathcal{L}) < \infty$ .*

**Proof:** One could argue that the Friedrich's extension of  $\mathcal{L}^\diamond$  is self-adjoint, after which, one will need to identify the domain as  $H^3(\mathbb{R})$ . We will instead follow a more direct route in constructing a self-adjoint extension of the symmetric operator  $\mathcal{L}^\diamond$ . To that end, let

$$\mathcal{L}^\diamond = |\partial_x|^{1/2} \mathcal{L} |\partial_x|^{1/2} = -|\partial_x| \partial_x^2 + |\partial_x|^{1/2} V |\partial_x|^{1/2} = |\partial_x|^3 + |\partial_x|^{1/2} V |\partial_x|^{1/2} =: \mathcal{L}_0^\diamond + \mathcal{K}^\diamond,$$

where  $D(\mathcal{L}_0^\diamond) = H^3(\mathbb{R})$ ,  $D(\mathcal{K}^\diamond) = H^1(\mathbb{R})$ . Clearly, with these assignments,  $\mathcal{L}_0^\diamond$  is self-adjoint, while  $\mathcal{K}^\diamond$  is a symmetric operator. Furthermore,  $\sigma(\mathcal{L}_0^\diamond) = \sigma_{\text{ess}}(\mathcal{L}_0^\diamond) = [0, \infty)$ .

According to the Weyl's essential spectrum theorem (and more specifically [17, Corollary 2, page 113]), we may conclude that  $\mathcal{L}^\diamond = \mathcal{L}_0^\diamond + \mathcal{K}^\diamond$  is self-adjoint and  $\sigma_{\text{ess}}(\mathcal{L}^\diamond) = \sigma_{\text{ess}}(\mathcal{L}_0^\diamond) = [0, \infty)$  provided we can establish that  $\mathcal{K}^\diamond$  is a relatively compact perturbation of  $\mathcal{L}_0^\diamond$ . This amounts to showing that

$$|\partial_x|^{1/2} V |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R}) \text{ is compact.}$$

Since it is clear that  $|\partial_x|^{1/2} V |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} \in B(L^2(\mathbb{R}))$ , it will suffice (by Relich's criteria) to establish

$$|\partial_x|^{1/2} V |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} : L^2(\mathbb{R}) \rightarrow \dot{H}^{1/2}(\mathbb{R}) \quad (3.3)$$

$$\| |\partial_x|^{1/2} V |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} f(x) \| \leq \frac{C}{\sqrt{|x|}} \|f\|_{L^2}, \quad |x| \gg 1. \quad (3.4)$$

For the proof of (3.3), we have

$$\begin{aligned} \| |\partial_x|^{1/2} V |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} f \|_{\dot{H}^{1/2}(\mathbb{R})} &= \| \partial_x [V |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} f] \|_{L^2(\mathbb{R})} \\ &\leq \|V'\|_{L^\infty} \| |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} f \|_{L^2} + \\ &\quad \|V\|_{L^\infty} \| |\partial_x|^{3/2} (i + |\partial_x|^3)^{-1} f \|_{L^2} \\ &\leq C(\|V'\|_{L^\infty} + \|V\|_{L^\infty}) \|f\|_{L^2}. \end{aligned}$$

For the proof of (3.4), first fix  $x$  with  $|x| \gg 1$ . Denoting  $h(x) = |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} f$  and  $G(x) = V(x)h(x)$ , write

$$\begin{aligned} |\partial_x|^{1/2} V |\partial_x|^{1/2} (i + |\partial_x|^3)^{-1} f(x) &= \sqrt{2\pi} \int |\xi|^{1/2} \hat{G}(\xi) e^{2\pi i x \xi} dx \\ &= \sqrt{2\pi} \sum_{k=-\infty}^{\infty} 2^{k/2} \int \tilde{\varphi}(2^{-k} \xi) \hat{G}(\xi) e^{2\pi i x \xi} dx, \end{aligned}$$

where  $\tilde{\varphi}(z) = |z|^{1/2} \varphi(z)$ . For the portion of the sum  $k$  with  $2^k < |x|^{-1}$ , we have

$$\sum_{k: 2^k < |x|^{-1}} 2^{k/2} \int \tilde{\varphi}(2^{-k} \xi) \hat{G}(\xi) e^{2\pi i x \xi} dx \leq C|x|^{-1/2} \sup_k \|G_{\sim 2^k}\|_{L^\infty} \leq C|x|^{-1/2} \|G\|_{L^\infty} \leq C|x|^{-1/2} \|V\|_{L^\infty} \|h\|_{L^\infty}.$$

By Sobolev embedding, for any  $m > 1/2$ ,  $\|h\|_{L^\infty} \leq C_m \|h\|_{H^m} \leq C \|f\|_{L^2}$ . Regarding the case of  $k$  with  $2^k \geq |x|^{-1}$  we integrate by parts to get

$$\int \tilde{\varphi}(2^{-k} \xi) \hat{G}(\xi) e^{2\pi i x \xi} dx = \frac{i}{2\pi x} \left( 2^{-k} \int \tilde{\varphi}'(2^{-k} \xi) \hat{G}(\xi) e^{2\pi i x \xi} dx + \int \tilde{\varphi}(2^{-k} \xi) \hat{G}'(\xi) e^{2\pi i x \xi} dx \right).$$

The first term is estimated via

$$\left| \sum_{k: 2^k \geq |x|^{-1}} \frac{2^{-k/2}}{2\pi x} \int \tilde{\varphi}'(2^{-k}\xi) \hat{G}(\xi) e^{2\pi i x \xi} dx \right| \leq C|x|^{-1/2} \sup_k \|G_{\sim 2^k}\|_{L^\infty},$$

whence the estimate finishes as the one a few lines up. Finally, upon introducing the smooth function  $\Phi(z) = \tilde{\varphi}(z)/z$ , rewrite

$$\int \tilde{\varphi}(2^{-k}\xi) \hat{G}'(\xi) e^{2\pi i x \xi} dx = 2^{-k} \int \Phi(2^{-k}\xi) [\xi \hat{G}'(\xi)] e^{2\pi i x \xi} dx,$$

so that

$$\left| \sum_{k: 2^k \geq |x|^{-1}} \frac{2^{-k/2}}{2\pi x} \int \Phi(2^{-k}\xi) [\xi \hat{G}'(\xi)] e^{2\pi i x \xi} dx \right| \leq C|x|^{-1/2} \sup_k \|H_{\sim 2^k}\|_{L^\infty} \leq C|x|^{-1/2} \|H\|_{L^\infty}$$

where  $H$  is defined through its Fourier transform,  $\hat{H}(\xi) := \xi \hat{G}'(\xi)$ . From the formula for  $H$  one can easily identify it; namely,  $H(x) = c_0 \partial_x(xV(x)h(x))$  for some constant  $c_0$ . Finally, by Hölder's inequality and Sobolev embedding we conclude with

$$\|H\|_{L^\infty} \leq C(\|xV(x)\|_{L^\infty} + \|xV'(x)\|_{L^\infty} + \|V\|_{L^\infty})(\|h\|_{L^\infty} + \|h'\|_{L^\infty}) \leq C_V \|f\|_{L^2}. \quad \square$$

#### 4. THE HAMILTONIAN-KREIN INDEX THEOREM

We are now ready to apply the instability index formula of [11, 12]. For the reformulated eigenvalue problem (3.2) let  $k_r$  denote the number of real-valued and positive eigenvalues (counting multiplicity), and let  $k_c$  be the number of complex-valued eigenvalues with positive real part. Since the imaginary part of  $\mathcal{L}$  satisfies  $\text{Im}(\mathcal{L}) = 0$ , it is the case that  $k_c$  is an even integer. Finally, for (potentially embedded) purely imaginary eigenvalues  $\lambda \in i\mathbb{R}$ , let  $E_\lambda$  denote the corresponding eigenspace. The negative Krein index of the eigenvalue is given by

$$k_i^-(\lambda) = n(\langle \mathcal{L}^\diamond|_{E_\lambda} u, u \rangle),$$

and the total negative Krein index is given by

$$k_i^- = \sum_{\lambda \in i\mathbb{R}} k_i^-(\lambda).$$

Here we use the notation  $S|_E = P_E S P_E$ , where  $P_E$  is the orthogonal projection onto the subspace  $E$ . It is the case that  $k_i^-$  is also an even integer. The Hamiltonian-Krein index is defined by

$$K_{\text{Ham}} := k_r + k_c + k_i^-$$

the index counts the total number of (potentially) unstable eigenvalues.

*Remark 4.1.* The negative Krein index of the operator is defined in terms of the the eigenvalue problem (3.2). Using the definition of  $\mathcal{L}^\diamond$  it can be rewritten as

$$k_i^-(\lambda) = n(\langle \mathcal{L}|_{E_\lambda} |\partial_x|^{1/2} u, |\partial_x|^{1/2} u \rangle).$$

In terms of the original eigenvalue problem (1.1), upon using the transformation that moved the first to the second yields that in terms of the original variables, the negative Krein index can be rewritten as

$$k_i^-(\lambda) = n(\langle \mathcal{L}|_{E_\lambda} u, u \rangle).$$

This is the expected definition, and the one that would have been used if the operator  $\partial_x$  had bounded inverse.

For the eigenvalue problem (3.2) the previous index theory relates  $K_{\text{Ham}}$  to the finite number  $n(\mathcal{L}^\diamond|_{S^\perp})$ , where  $S$  is some finite-dimensional subspace. In order to apply this theory, however, it is not only necessary that  $\mathcal{J}$  have bounded inverse, but that  $\sigma_{\text{ess}}(\mathcal{L}^\diamond)$  be uniformly bounded away from the origin. Unfortunately, this technical assumption is no longer necessarily valid, as in the applications it will generically be the case that the essential spectrum touches the origin, i.e.,  $\sigma_{\text{ess}}(\mathcal{L}^\diamond) = [0, +\infty)$  (e.g., see Proposition 3.5). We overcome this technical difficulty with the following argument.

For  $0 < \epsilon \ll 1$  consider the sandwiched operator

$$\mathcal{L}_\epsilon^\diamond := (-\partial_x^2 + \epsilon^2)^{1/4} \mathcal{L} (-\partial_x^2 + \epsilon^2)^{1/4}.$$

Note that  $\mathcal{L}^\diamond = \mathcal{L}_0^\diamond$ , and moreover  $\mathcal{L}_\epsilon^\diamond \rightarrow \mathcal{L}^\diamond$  as  $\epsilon \rightarrow 0^+$  in the weak operator topology. That is, for each pair of test functions  $\chi, \psi$ ,

$$\lim_{\epsilon \rightarrow 0} \langle \mathcal{L}_\epsilon^\diamond \chi, \psi \rangle = \langle \mathcal{L}^\diamond \chi, \psi \rangle \quad (4.1)$$

Since the operator  $(-\partial_x^2 + \epsilon^2)^{1/4}$  is invertible for any  $\epsilon > 0$ ,<sup>7</sup>

$$n(\mathcal{L}_\epsilon^\diamond) = n(\mathcal{L});$$

thus, by using Lemma 3.4 we have the double equality

$$n(\mathcal{L}_\epsilon^\diamond) = n(\mathcal{L}) = n(\mathcal{L}^\diamond). \quad (4.2)$$

Regarding the essential spectrum, we would like to say that it is pushed off the origin and becomes  $\sigma_{\text{ess}}(\mathcal{L}_\epsilon^\diamond) \subseteq [\kappa^2 \epsilon, +\infty)$ . This is indeed the case, but it needs some justification.

**Proposition 4.2.** *The essential spectrum of the sandwiched operator satisfies*

$$\sigma_{\text{ess}}(\mathcal{L}_\epsilon^\diamond) \subseteq [\kappa^2 \epsilon, +\infty).$$

**Proof:** Recall that by assumption  $\mathcal{L} = \mathcal{L}_0 + \mathcal{K}$ , where  $\mathcal{K}$  is relatively compact perturbation of  $\mathcal{L}_0 \geq \kappa^2 \mathcal{I}$ . In particular,  $\mathcal{L}_0$  is invertible and  $\mathcal{K}\mathcal{L}_0^{-1}$  is a compact operator. Denote

$$\tilde{\mathcal{L}}_\epsilon = (-\partial_x^2 + \epsilon^2)^{1/4} \mathcal{L}_0 (-\partial_x^2 + \epsilon^2)^{1/4}; \quad \tilde{\mathcal{K}}_\epsilon = (-\partial_x^2 + \epsilon^2)^{1/4} \mathcal{K} (-\partial_x^2 + \epsilon^2)^{1/4},$$

so that  $\mathcal{L}_\epsilon^\diamond = \tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon$ . By applying the Fourier transform, we see that

$$\sigma(\tilde{\mathcal{L}}_\epsilon) = \sigma_{\text{ess}}(\tilde{\mathcal{L}}_\epsilon) = \text{Range}[\xi \mapsto (4\pi^2 \xi^2 + \epsilon^2)^{1/4} q(\xi) (4\pi^2 \xi^2 + \epsilon^2)^{1/4}] \subseteq [\kappa^2 \epsilon, \infty),$$

since by assumption  $q(\xi) \geq \kappa^2$ . Thus, it remains to show

$$\sigma_{\text{ess}}(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon) = \sigma_{\text{ess}}(\tilde{\mathcal{L}}_\epsilon). \quad (4.3)$$

In order to show (4.3), we will verify that  $(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} - (\tilde{\mathcal{L}}_\epsilon + i)^{-1}$  is a compact operator, whence the result will follow from a standard lemma in spectral theory [17, Corollary 1, p. 113]. Indeed,

$$\tilde{\mathcal{L}}_\epsilon^{-1} \tilde{\mathcal{K}}_\epsilon \tilde{\mathcal{L}}_\epsilon^{-1} = (-\partial_x^2 + \epsilon^2)^{-1/4} \mathcal{L}_0^{-1} \mathcal{K} \mathcal{L}_0^{-1} (-\partial_x^2 + \epsilon^2)^{-1/4}$$

is compact, since  $\mathcal{K}\mathcal{L}_0^{-1}$  is compact by assumption. A multiple application of the resolvent identity yields the formula

$$(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} \tilde{\mathcal{K}}_\epsilon (\tilde{\mathcal{L}}_\epsilon + i)^{-1} = [(\tilde{\mathcal{L}}_\epsilon^{-1} - (\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} (\tilde{\mathcal{K}}_\epsilon + i) \tilde{\mathcal{L}}_\epsilon^{-1}) \tilde{\mathcal{K}}_\epsilon [(\tilde{\mathcal{L}}_\epsilon^{-1} + i \tilde{\mathcal{L}}_\epsilon^{-1} (\tilde{\mathcal{L}}_\epsilon + i)^{-1})],$$

which implies that  $(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} \tilde{\mathcal{K}}_\epsilon (\tilde{\mathcal{L}}_\epsilon + i)^{-1}$  is compact as well. From the resolvent identity again,

$$(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} - (\tilde{\mathcal{L}}_\epsilon + i)^{-1} = -(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} \tilde{\mathcal{K}}_\epsilon (\tilde{\mathcal{L}}_\epsilon + i)^{-1},$$

which implies the compactness of  $(\tilde{\mathcal{L}}_\epsilon + \tilde{\mathcal{K}}_\epsilon + i)^{-1} - (\tilde{\mathcal{L}}_\epsilon + i)^{-1}$ , and hence (4.3) is established.  $\square$

<sup>7</sup>In fact, one might argue as in Lemma 3.4 that for each  $\epsilon > 0$ ,  $n(\mathcal{L}_\epsilon^\diamond) = n(\mathcal{L})$

For Hamiltonian eigenvalue problems the Hamiltonian-Krein index theory can be applied if the skew-operator has a bounded inverse, and if the self-adjoint operator (a) has a finite number of negative eigenvalues, and (b) has an essential spectrum which is bounded away from the origin. Thus, this theory is applicable for the operator  $\mathcal{J}\mathcal{L}_\epsilon^\diamond$  for any  $\epsilon > 0$ . We now wish to establish the index for the one-parameter family of eigenvalue problems given by

$$\mathcal{J}\mathcal{L}_\epsilon^\diamond u = \lambda u \quad \Rightarrow \quad \mathcal{L}_\epsilon^\diamond u = \lambda \mathcal{J}^{-1}u. \quad (4.4)$$

Afterwards, we will use a limiting argument to establish the index for the original problem (1.1).

Since  $\ker(\mathcal{L}_\epsilon^\diamond) = \text{span}\{(-\partial_x^2 + \epsilon^2)^{-1/4}\psi_0\}$ , and since  $\mathcal{L}_\epsilon^\diamond$  is self-adjoint, for  $\lambda \neq 0$  the eigenvalue problem (4.4) can by the Fredholm alternative have a solution if and only if

$$\langle \mathcal{J}^{-1}u, (-\partial_x^2 + \epsilon^2)^{-1/4}\psi_0 \rangle = 0 \quad \Rightarrow \quad \langle u, (-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0 \rangle = 0.$$

Thus, upon setting  $S_\epsilon := \text{span}\{(-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0\}$ , the search for nonzero eigenvalues is accomplished by considering the constrained problem

$$\mathcal{J}\mathcal{L}_\epsilon^\diamond u = \lambda u, \quad u \in S_\epsilon^\perp. \quad (4.5)$$

As a consequence of [11, 12] it is true that for  $\epsilon > 0$  the Hamiltonian-Krein index associated with the eigenvalue problem (4.5) satisfies

$$K_{\text{Ham}}^\epsilon = n(\mathcal{L}_\epsilon^\diamond|_{S_\epsilon^\perp}).$$

Using, e.g., [10], we have that the negative index on the right satisfies

$$\begin{aligned} n(\mathcal{L}_\epsilon^\diamond|_{S_\epsilon^\perp}) &= n(\mathcal{L}_\epsilon^\diamond) - n(\langle (\mathcal{L}_\epsilon^\diamond)^{-1}(-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0, (-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0 \rangle) \\ &= n(\mathcal{L}) - n(\langle (\mathcal{L}^\diamond)^{-1}(-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0, (-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0 \rangle). \end{aligned}$$

The second equality follows from (4.2). Regarding the second quantity on the right, by the definition of  $\mathcal{L}_\epsilon^\diamond$  we have

$$\begin{aligned} \langle (\mathcal{L}_\epsilon^\diamond)^{-1}(-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0, (-\partial_x^2 + \epsilon^2)^{-1/4}|\partial_x|\partial_x^{-1}\psi_0 \rangle &= \\ \langle \mathcal{L}^{-1}(-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0, (-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0 \rangle. \end{aligned}$$

Note that the expression  $\mathcal{L}^{-1}[(-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0]$  makes sense, since  $(-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0 \in \ker(\mathcal{L})^\perp$ . Since  $(-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0 \rightarrow \partial_x^{-1}\psi_0$  in  $L^2(\mathbb{R})$  sense, we have

$$\langle \mathcal{L}^{-1}(-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0, (-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0 \rangle \rightarrow \langle \mathcal{L}^{-1}\partial_x^{-1}\psi_0, \partial_x^{-1}\psi_0 \rangle.$$

It follows that for all  $\epsilon > 0$  sufficiently small

$$n(\langle \mathcal{L}^{-1}(-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0, (-\partial_x^2 + \epsilon^2)^{-1/2}|\partial_x|\partial_x^{-1}\psi_0 \rangle) = n(\langle \mathcal{L}^{-1}(\partial_x^{-1}\psi_0), \partial_x^{-1}\psi_0 \rangle).$$

Putting this all together, we see that the Hamiltonian-Krein index for  $\epsilon > 0$  small satisfies

$$K_{\text{Ham}}^\epsilon = n(\mathcal{L}) - n(\langle \mathcal{L}^{-1}(\partial_x^{-1}\psi_0), \partial_x^{-1}\psi_0 \rangle).$$

Since the quantity on the right is  $\epsilon$ -independent, and since the index is integer-valued, we can take the limit as  $\epsilon \rightarrow 0^+$  and conclude that for the original eigenvalue problem (3.2),

$$K_{\text{Ham}}^0 = n(\mathcal{L}) - n(\langle \mathcal{L}^{-1}(\partial_x^{-1}\psi_0), \partial_x^{-1}\psi_0 \rangle).$$

In other words, the Hamiltonian-Krein index for the equivalent problem (3.2) does not depend at all on the reformulation of the eigenvalue problem, and can be stated in terms of the original operators  $\partial_x$  and  $\mathcal{L}$ .

Finally, recall that the reformulated eigenvalue problem assumed that all of the eigenfunctions resided in  $\dot{H}^{-1/2}(\mathbb{R})$ . Since the eigenfunctions associated with nonzero eigenvalues must reside in  $\dot{H}^{-1}(\mathbb{R})$ , the index for the original eigenvalue problem (1.1) must be the same as for the reformulated problem. This allows us to conclude with the following:

**Theorem 4.3.** Consider the eigenvalue problem

$$\partial_x \mathcal{L}u = \lambda u, \quad u \in L^2(\mathbb{R}),$$

where the self-adjoint operator  $\mathcal{L}$  satisfies  $D(\mathcal{L}) = H^s(\mathbb{R})$  for some  $s \geq 0$ . Assuming that

$$\langle \mathcal{L}^{-1}(\partial_x^{-1} \psi_0), \partial_x^{-1} \psi_0 \rangle \neq 0,$$

the Hamiltonian-Krein index satisfies

$$K_{\text{Ham}} = n(\mathcal{L}) - n(\langle \mathcal{L}^{-1}(\partial_x^{-1} \psi_0), \partial_x^{-1} \psi_0 \rangle).$$

*Remark 4.4.* The result of [Theorem 4.3](#) is the one that would be expected if the skew operator  $\partial_x$  had a bounded inverse. On the other hand, when considering the eigenvalue problem on the space of spatially periodic functions, the index satisfies

$$K_{\text{Ham}} = n(\mathcal{L}) - n(D), \quad D = \begin{pmatrix} \langle \mathcal{L}^{-1}(\partial_x^{-1} \psi_0), \partial_x^{-1} \psi_0 \rangle & \langle \mathcal{L}^{-1}(\partial_x^{-1} \psi_0), 1 \rangle \\ \langle \mathcal{L}^{-1}(\partial_x^{-1} \psi_0), 1 \rangle & \langle \mathcal{L}^{-1}(1), 1 \rangle \end{pmatrix}$$

[5, 6]. The (1,1) term in the matrix  $D$  is exactly that as seen in the above theorem. In this latter case, the additional terms in  $D$  arise from the fact that  $\ker(\partial_x) = \text{span}\{1\}$ : this kernel is not present when considering the problem on the whole line in the space  $L^2(\mathbb{R})$ . In both cases the eigenfunctions associated with nonzero eigenvalues must have mean zero, but interestingly enough it is only in the latter case that this restriction actually has an effect on the index.

## 5. APPLICATIONS TO KdV-LIKE PROBLEMS

Consider an evolution equation written in the form

$$\partial_t u = \partial_x \mathcal{H}'(u), \tag{5.1}$$

where  $\mathcal{H}(u) : H^\ell(\mathbb{R}) \mapsto \mathbb{R}$  is some smooth energy functional. It is assumed that solutions are invariant under spatial translation, and that there are no other symmetries present in the system. In traveling coordinates  $\xi = x - ct$  the equation can be rewritten as

$$\partial_t u = \partial_\xi [\mathcal{H}'(u) + cu].$$

For a given  $a \in \mathbb{R}$  the solitary wave solutions  $U_c$  will satisfy

$$\mathcal{H}'(U_c) = -cU_c + a. \tag{5.2}$$

It will be assumed that these solutions are smooth in the wave-speed over some nonempty interval. It will further be assumed that  $U_c \in H^k(\mathbb{R})$  for some  $k \geq 1$  and all  $c$ .

The linearized eigenvalue problem is given by

$$\partial_\xi \mathcal{L}u = \lambda u, \quad \mathcal{L} := \mathcal{H}''(U_c) + c. \tag{5.3}$$

Since solutions are invariant under spatial translation, it is the case that

$$\mathcal{L}(\partial_\xi U_c) = 0.$$

By assumption it is clearly the case that  $\partial_\xi U_c \in \dot{H}^{-1}(\mathbb{R})$ . Now, differentiating the existence problem (5.2) with respect to  $c$  yields

$$\mathcal{L}(\partial_c U_c) = -U_c \quad \Rightarrow \quad \partial_c U_c = -\mathcal{L}^{-1}(U_c).$$

As a consequence, upon making the equivalence that  $\psi_0 = \partial_\xi U_c$  in the statement of [Theorem 4.3](#), it is seen that  $\partial_\xi^{-1} \psi_0 = U_c$  and hence

$$\langle \mathcal{L}^{-1} \partial_\xi^{-1} \psi_0, \partial_\xi^{-1} \psi_0 \rangle = -\langle \partial_c U_c, U_c \rangle = -\frac{1}{2} \partial_c \langle U_c, U_c \rangle.$$

Defining  $p(a) = 1$  for  $a > 0$  and  $p(a) = 0$  for  $a < 0$ , the Hamiltonian-Krein index for the eigenvalue problem (5.3) can then be stated as:

**Theorem 5.1.** Consider the KdV-like evolution equation (5.1). Let  $U_c$  be a solitary wave which solves (5.2) and satisfies the properties proscribed above. Let

$$\mathcal{L} := \mathcal{H}''(U_c) + c$$

be the linearization about the wave in traveling coordinates. Assume that

$$\ker(\mathcal{L}) = \text{span}\{\partial_\xi U_c\}, \quad \partial_c \langle U_c, U_c \rangle \neq 0.$$

The Hamiltonian-Krein index for the linearized eigenvalue problem (5.3) is then

$$K_{\text{Ham}} = n(\mathcal{L}) - p(\partial_c \langle U_c, U_c \rangle).$$

*Remark 5.2.* Since  $k_c$  and  $k_1^-$  are even integers, the underlying wave will be spectrally unstable with  $k_r \geq 1$  if  $K_{\text{Ham}}$  is odd.

*Remark 5.3.* From Theorem 5.1 we (partially) recover the following well-known result. Suppose that  $n(\mathcal{L}) = 1$ . If  $\partial_c \langle U_c, U_c \rangle > 0$ , then under some additional genericity conditions it is known that the wave is orbitally stable [4] (also see [9, Chapter 5]). As we see from Theorem 5.1, under this scenario  $K_{\text{Ham}} = 0$ , so that all of the spectra is purely imaginary, and any (embedded) eigenvalues must have positive Krein signature. On the other hand, if  $\partial_c \langle U_c, U_c \rangle < 0$ , then  $K_{\text{Ham}} = 1$  implies that  $k_r = 1$ , which corroborates the Evans function calculation of Pego and Weinstein [15, 16].

### 5.1. Fractional KdV equations

The exact form of  $\mathcal{L}$  in Theorem 5.1 is unimportant, as long as the desired properties hold. Thus, as a generalization of the above we obtain the following result of Lin [14, Theorem 1] concerning the more general KdV-type equation

$$\partial_t u - \partial_x (\mathcal{M}u - f(u)) = 0, \tag{5.4}$$

where  $f \in C^1$ ,  $f(0) = f'(0) = 0$ , and the operator  $\mathcal{M}$  is defined through its Fourier symbol via  $\widehat{\mathcal{M}g}(\xi) = \alpha(\xi)\hat{g}(\xi)$ . Examples of this type include the KdV equations (and its generalizations) with  $\mathcal{M} = -\partial_x^2$ , the Benjamin-Ono equation with  $\mathcal{M} = |\partial_x|$ , etc. We henceforth will assume that the multiplier  $\alpha(\xi)$  is a continuous function of its argument with  $\lim_{|\xi| \rightarrow \infty} \alpha(\xi) = \infty$ .

**Corollary 5.4.** For the generalized KdV-type equation (5.4) assume that the linearized operator  $\mathcal{L} = \mathcal{M} + c - f'(U_c)$  satisfies the original assumptions (a), (b), (c) with

$$\ker(\mathcal{L}) = \{\partial_\xi U_c\}.$$

The Hamiltonian-Krein index for the spectral problem  $\partial_\xi \mathcal{L}u = \lambda u$  is

$$K_{\text{Ham}} = n(\mathcal{L}) - p(\partial_c \langle U_c, U_c \rangle). \tag{5.5}$$

In particular,

- the wave  $U_c$  is spectrally unstable if  $K_{\text{Ham}}$  is odd, i.e.,
  - $n(\mathcal{L})$  is even and  $\partial_c \langle U_c, U_c \rangle > 0$
  - $n(\mathcal{L})$  is odd and  $\partial_c \langle U_c, U_c \rangle < 0$
- the wave  $U_c$  is spectrally stable if  $n(\mathcal{L}) = 1$  and  $\partial_c \langle U_c, U_c \rangle > 0$ .

One should compare these results with the corresponding results of Lin [14, Theorem 1] for KdV-type equations. In particular, we do not require the multiplier  $\alpha(\xi)$  to have any specific form, like  $\alpha(\xi) \sim |\xi|^m$ , but only the natural conditions

- $\mathcal{M} + c \geq \delta^2 \mathcal{I} > 0$  for some  $\delta > 0$ ,

- $f'(U_c)$  is a relatively compact perturbation of  $\mathcal{M} + c$ .

The condition  $\mathcal{M} + c \geq \delta^2 \mathcal{I}$  is satisfied by simply requiring that  $\alpha(\xi) \geq 0$  and  $c > 0$ , in which case, we may select  $\delta = c/2$ . The relative compactness of  $f'(U_c)$  follows from *any (even power) decay* of  $U_c$  at infinity, as well as *any (even power) decay* of the kernel of  $(\mathcal{M} + c)^{-1}$ , which is given by

$$K(x) = \int_{-\infty}^{+\infty} \frac{1}{c + \alpha(\xi)} e^{2\pi i \xi x} d\xi.$$

This last condition is satisfied, under very mild growth requirements of  $\alpha$ . To see this, an easy integration by parts argument implies

$$|K(x)| \leq \frac{C}{|x|} \int_{-\infty}^{\infty} \frac{|\alpha'(\xi)|}{(c + \alpha(\xi))^2} d\xi$$

This allows us to formulate more specific and easily verifiable conditions which imply that the essential spectrum assumption (b) is satisfied:

**Proposition 5.5.** *Let  $c > 0$  and the wave  $U_c$  and the multiplier  $\alpha$  satisfy*

- $U_c$  has some power decay at  $\infty$ ,
- the Fourier multiplier  $\alpha(\xi)$ 
  - is a continuous differentiable a.e. function with  $\alpha(\xi) \geq 0$  and  $\lim_{|\xi| \rightarrow \infty} \alpha(\xi) = \infty$ ,
  - satisfies the estimate

$$\int_{-\infty}^{+\infty} \frac{|\alpha'(\xi)|}{(c + \alpha(\xi))^2} d\xi < \infty. \quad (5.6)$$

Then  $\mathcal{M} + c > \delta^2 \mathcal{I}$ , and  $f'(U_c)$  is a relatively compact perturbation of  $\mathcal{M} + c$ . In other words, requirement (b) for the operator  $\mathcal{L} = \mathcal{M} + c - f'(U_c)$  is satisfied.

It is instructive to consider the spectral stability of the traveling wave  $U_c(x) = 4c/(1 + c^2 x^2)$  of the Benjamin-Ono equation

$$\partial_t u - \partial_x (|\partial_x| u - u^2) = 0.$$

The orbital stability of these waves has already been established by Albert and Bona [1] and Albert et al. [2]; hence, this wave is spectrally stable. An alternative approach is to use the approach of Corollary 5.4 and use the spectral information provided by the work of Amick and Toland [3], see the remarks after Theorem 5.6.

In fact, we have more general result, which is applicable to the so-called fractional KdV equations (or, fractional Benjamin-Ono equation as they are referred to in [7])

$$\partial_t u - \partial_x (|\partial_x|^s u - u^{p+1}) = 0. \quad (5.7)$$

Using the traveling wave ansatz  $U_c(\xi)$ ,  $\xi = x - ct$  with  $c > 0$  in (5.7) (and requiring that  $U_c$  vanishes at  $\pm\infty$ ) leads us to the existence equation

$$|\partial_\xi|^s U_c + c U_c - U_c^{p+1} = 0 \quad (5.8)$$

An elementary scaling analysis then shows that  $U_c(\xi) = c^{1/p} Q(c^{1/s} \xi)$ , where  $Q$  satisfies

$$|\partial_\xi|^s Q + Q - Q^{p+1} = 0. \quad (5.9)$$

Assume that  $0 < s < 2$ , and set

$$p_{\max}(s) = \begin{cases} 2s/(1-s), & 0 < s < 1 \\ +\infty, & 1 \leq s < 2. \end{cases}$$

It was recently shown by Frank and Lenzmann [7] that if  $0 < p < p_{\max}(s)$ , then (5.9) has an unique (up to translation) ground state solution  $Q$  which is positive and bell-shaped, i.e., even and decreasing in  $(0, \infty)$ .



In addition, the solution  $Q$  has decay as  $\xi \rightarrow +\infty$ ; in fact,  $|Q(\xi)| \leq C|\xi|^{-1}$  (see [7, Lemma 5.6]). Furthermore, its linearized operator,

$$\mathcal{L}_+ = |\partial_\xi|^s + 1 - (p+1)Q^p,$$

satisfies  $n(\mathcal{L}_+) = 1$ , the kernel is simple with  $\ker(\mathcal{L}_+) = \text{span}\{\partial_\xi Q\}$ , and the rest of the spectrum is positive, with a spectral gap at the zero. A simple rescaling then implies that similar statements hold for the linearized operator

$$\mathcal{L}_c = |\partial_\xi|^s + c - (p+1)U_c^p.$$

Going back to the linearized stability of  $U_c$ , we see that we need to consider the eigenvalue problem

$$\partial_\xi \mathcal{L}_c z = \lambda z.$$

By the decay of  $Q$ , the relative compactness of  $U_c^p$  follows easily (indeed, note that the condition is satisfied for when the multiplier is  $\alpha(\xi) = |\xi|^s$  for any  $s > 0$ ). Thus, we can apply Corollary 5.4 to conclude that

$$K_{\text{Ham}} = 1 - p(\partial_c \langle U_c, U_c \rangle).$$

The stability of the traveling wave  $U_c$  is equivalent to the positivity of  $\partial_c \langle U_c, U_c \rangle$ . Since

$$\partial_c \langle U_c, U_c \rangle = \text{const.} \left( \frac{2}{p} - \frac{1}{s} \right) c^{\frac{2}{p} - \frac{1}{s} - 1}$$

for some positive constant, we have shown the following:

**Theorem 5.6.** *Let  $0 < s < 2$  and  $0 < p < p_{\max}$ . The unique ground state traveling wave solutions  $U_c$  are spectrally stable if*

$$0 < p < 2s.$$

*Otherwise, if  $2s < p < p_{\max}(s)$  there is precisely one positive real eigenvalue, and the rest of the spectrum is purely imaginary.*

*Remark 5.7.* The Benjamin-Ono case corresponds to the case  $s = 1, p = 2$ , which is the borderline case in the above computation. Since in this case  $\partial_c \langle U_c, U_c \rangle = 0$ , the generalized kernel for  $\partial_\xi \mathcal{L}_c$  will have dimension of at least three. The calculation of the Hamiltonian-Krein index would have to be modified in order to take into account the fact that the dimension of the generalized kernel is larger than two. This is a relatively straightforward exercise which we will leave for the interested reader (e.g., see [10, Index Theorem 2.1] and the references therein for an indication as to what modifications must be made). The end result is the expected one: the index satisfies  $K_{\text{Ham}} = 0$ .

*Remark 5.8.* The result of Theorem 5.6 recovers the classical KdV results for the limiting case  $s = 2$ , which does not require the theory of [7] in order to conclude the existence of the wave  $U_c$ . Here we then get the well-known result of spectral (which in this case implies orbital) stability for ground state solutions to the generalized KdV whenever  $p < 4$ , and spectral instability for  $p > 4$ .

## 6. APPLICATIONS TO BBM-LIKE PROBLEMS

Consider the BBM-type problem

$$(\mathcal{I} + \mathcal{M})\partial_t u + \partial_x(u + f(u)) = 0, \tag{6.1}$$

under the same conditions on the nonlinearity  $f$  and the dispersion operator  $\mathcal{M}$  as in Corollary 5.4. These problems have been considered in this generality by numerous authors; for example, see the paper [14] for an extensive discussion. In addition to the requirements on  $\mathcal{M}$  presented in the discussion leading to Corollary 5.4, we require that  $\mathcal{I} + \mathcal{M} \geq \delta^2 \mathcal{I} > 0$ , which hence possesses a square root and a bounded inverse<sup>8</sup>. This last requirement, in view of the form required of  $\mathcal{M}$  amounts to the multiplier satisfying the inequality  $1 + \alpha(\xi) \geq \delta^2$ .

<sup>8</sup>In the classical BBM model  $\mathcal{M} = -\partial_x^2$ , in which case the invertibility of  $(1 - \partial_x^2)^{-1}$  is a well-settled issue



After integration in the traveling variable  $\xi = x - ct$  the traveling wave solution  $U_c$ <sup>9</sup> will satisfy the elliptic PDE

$$c\mathcal{M}U_c(\xi) + (c-1)U_c(\xi) - f(U_c) = 0. \quad (6.2)$$

Introduce the corresponding linearized operator

$$\mathcal{L}_0 := c\mathcal{M} + (c-1) - f'(U_c).$$

Clearly, a differentiation<sup>10</sup> in  $\xi$  of the defining equation (6.2) yields that  $\mathcal{L}_0(\partial_\xi U_c) = 0$ : we henceforth assume that  $\ker(\mathcal{L}_0) = \text{span}\{\partial_\xi U_c\}$ . Writing  $u = U_c(\xi) + v(\xi, t)$  in (6.1) and linearizing yields

$$(\mathcal{I} + \mathcal{M})\partial_t v = \partial_\xi [c\mathcal{M} + (c-1) - f'(U_c)]v = \partial_\xi \mathcal{L}_0 v,$$

which in turn yields the eigenvalue problem

$$\partial_\xi \mathcal{L}_0 v = \lambda(\mathcal{I} + \mathcal{M})v. \quad (6.3)$$

In order to put this linearized problem in the desired framework, we introduce the variable

$$z := (\mathcal{I} + \mathcal{M})^{1/2} v.$$

Taking  $(\mathcal{I} + \mathcal{M})^{-1/2}$  on both sides of (6.3) yields the new eigenvalue problem

$$\partial_\xi [(\mathcal{I} + \mathcal{M})^{-1/2} \mathcal{L}_0 (\mathcal{I} + \mathcal{M})^{-1/2}] z = \lambda z. \quad (6.4)$$

If  $\lambda$  is an eigenvalue for (6.4) with corresponding eigenfunction  $z$ , then the fact that  $(\mathcal{I} + \mathcal{M})^{-1/2}$  is a bounded operator implies that  $\lambda$  is an eigenvalue for (6.3) with corresponding eigenfunction  $v = (\mathcal{I} + \mathcal{M})^{-1/2} z$ . On the other hand, suppose that  $\lambda$  is a nonzero eigenvalue for (6.3) with corresponding eigenfunction  $v$ . Since  $(\mathcal{I} + \mathcal{M})v$  necessarily makes sense, it is true that  $z = (\mathcal{I} + \mathcal{M})^{1/2} v$  is well-defined; thus,  $\lambda$  is also an eigenvalue for (6.4). In conclusion, when looking for nonzero eigenvalues the two spectral problems are equivalent. Denote

$$\mathcal{L} := (\mathcal{I} + \mathcal{M})^{-1/2} \mathcal{L}_0 (\mathcal{I} + \mathcal{M})^{-1/2},$$

so that the eigenvalue problem associated with (6.4) is of the desired form (1.1). We now discuss abstract conditions that would imply

- (a)  $\sigma_{\text{ess}}(\mathcal{L}) \subset [\kappa_0^2, \infty)$ ,  $\kappa_0 > 0$
- (b)  $n(\mathcal{L}) = n(\mathcal{L}_0)$ .

The condition (b) is satisfied because  $(\mathcal{I} + \mathcal{M})^{-1/2}$  is a bounded and symmetric operator. Regarding (a), write

$$\mathcal{L} = (\mathcal{I} + \mathcal{M})^{-1/2} [c\mathcal{M} + (c-1)] (\mathcal{I} + \mathcal{M})^{-1/2} - (\mathcal{I} + \mathcal{M})^{-1/2} f'(U_c) (\mathcal{I} + \mathcal{M})^{-1/2}.$$

If  $(\mathcal{I} + \mathcal{M})^{-1/2} f'(U_c) (\mathcal{I} + \mathcal{M})^{-1/2}$  is a compact operator, then

$$\sigma_{\text{ess}}(\mathcal{L}) = \sigma_{\text{ess}}((\mathcal{I} + \mathcal{M})^{-1/2} [c\mathcal{M} + (c-1)] (\mathcal{I} + \mathcal{M})^{-1/2}) = \text{Range}[\xi \mapsto \frac{c\alpha(\xi) + c-1}{1 + \alpha(\xi)}] \subset [\kappa_0^2, \infty).$$

Indeed, the last inclusion holds because of the continuity of the symbol  $\frac{c\alpha(\xi) + c-1}{1 + \alpha(\xi)} \geq 0$ , the fact that  $c\alpha(\xi) + c-1 \geq \delta^2$ , and the limiting behavior  $\lim_{\xi \rightarrow \infty} \frac{c\alpha(\xi) + c-1}{1 + \alpha(\xi)} = c > 0$ . On the other hand, the compactness of the operator  $(\mathcal{I} + \mathcal{M})^{-1/2} f'(U_c) (\mathcal{I} + \mathcal{M})^{-1/2}$  holds under minimal assumptions on the decay of  $U_c$  and the kernel of  $(\mathcal{I} + \mathcal{M})^{-1}$ , see Proposition 5.5. More precisely, to show that the kernel  $K$  of  $(\mathcal{I} + \mathcal{M})^{-1/2}$  satisfies  $|K(x)| \leq C|x|^{-1}$ , we need that the multiplier  $\alpha(\xi)$  satisfies

$$\int_{-\infty}^{\infty} \frac{|\alpha'(\xi)|}{(1 + \alpha(\xi))^{3/2}} d\xi < \infty.$$

<sup>9</sup>which is henceforth assumed to be homoclinic to zero at  $\xi = \pm\infty$  and in addition, it has at least a power decay

<sup>10</sup>Here is where matters that  $\mathcal{M}$  is given by multiplier, so that  $\partial_\xi \mathcal{M} = \mathcal{M} \partial_\xi$

This is certainly satisfied in the cases of interest,  $\alpha(\xi) = |\xi|^s, s > 0$ .

Continuing with the analysis of the operator  $\mathcal{L}$ , we verify by a direct inspection

$$\mathcal{L}[(\mathcal{I} + \mathcal{M})^{1/2} \partial_\xi U_c] = (\mathcal{I} + \mathcal{M})^{-1/2} (\mathcal{L}_0[\partial_\xi U_c]) = 0,$$

which gives a definitive relationship between the kernels of the two operators. In particular, they have the same dimension. Regarding the Hamiltonian-Krein index for (6.4), the fact that  $(\mathcal{I} + \mathcal{M})^{1/2} \partial_\xi U_c = \partial_\xi (\mathcal{I} + \mathcal{M})^{1/2} U_c$  allows us to write

$$\langle \mathcal{L}^{-1} \partial_\xi^{-1} [(\mathcal{I} + \mathcal{M})^{1/2} \partial_\xi U_c], \partial_\xi^{-1} (\mathcal{I} + \mathcal{M})^{1/2} \partial_\xi U_c \rangle = \langle \mathcal{L}^{-1} [(\mathcal{I} + \mathcal{M})^{1/2} U_c], (\mathcal{I} + \mathcal{M})^{1/2} U_c \rangle,$$

which in terms of the original operator yields

$$\langle \mathcal{L}^{-1} [(\mathcal{I} + \mathcal{M})^{1/2} U_c], (\mathcal{I} + \mathcal{M})^{1/2} U_c \rangle = \langle \mathcal{L}_0^{-1} [(\mathcal{I} + \mathcal{M}) U_c], (\mathcal{I} + \mathcal{M}) U_c \rangle.$$

Now, taking a derivative in the variable  $c$  in the existence equation (6.2) yields

$$\mathcal{L}_0[\partial_c U_c] = -(\mathcal{I} + \mathcal{M}) U_c \quad \Rightarrow \quad \mathcal{L}_0^{-1} [(\mathcal{I} + \mathcal{M}) U_c] = -\partial_c U_c,^{11}$$

so that

$$\langle \mathcal{L}_0^{-1} [(\mathcal{I} + \mathcal{M}) U_c], (\mathcal{I} + \mathcal{M}) U_c \rangle = -\frac{1}{2} \partial_c \langle (\mathcal{I} + \mathcal{M}) U_c, U_c \rangle.$$

Upon applying Theorem 4.3 we have now shown the following:

**Theorem 6.1.** *Consider the BBM-type equation (6.1), assume further that  $\mathcal{L}_0$  satisfies*

- $\alpha(\xi)$  is continuous,  $\lim_{|\xi| \rightarrow \infty} \alpha(\xi) = \infty$ , and  $c\alpha(\xi) + c - 1 \geq \delta^2 > 0$  for some  $\delta > 0$
- $(\mathcal{I} + \mathcal{M})^{-1/2} f'(U_c)(\mathcal{I} + \mathcal{M})^{-1/2}$  is a compact operator on  $L^2(\mathbb{R})$
- $\ker(\mathcal{L}_0) = \text{span}\{\partial_\xi U_c\}$ .

*The Hamiltonian-Krein index for the spectral problem (6.3) is*

$$K_{\text{Ham}} = n(\mathcal{L}_0) - p(\partial_c \langle (\mathcal{I} + \mathcal{M}) U_c, U_c \rangle).$$

*In particular,*

- *the wave  $U_c$  is spectrally unstable if  $K_{\text{Ham}}$  is odd, i.e.,*
  - $n(\mathcal{L})$  is even and  $\partial_c \langle (\mathcal{I} + \mathcal{M}) U_c, U_c \rangle > 0$
  - $n(\mathcal{L})$  is odd and  $\partial_c \langle (\mathcal{I} + \mathcal{M}) U_c, U_c \rangle < 0$
- *the wave  $U_c$  is spectrally stable if  $n(\mathcal{L}) = 1$  and  $\partial_c \langle (\mathcal{I} + \mathcal{M}) U_c, U_c \rangle > 0$ .*

*Remark 6.2.* Comparing these results with the corresponding results of Lin [14, Theorem 1], we see that regarding the requirements on the multiplier  $\alpha(\xi)$ , they are slightly more general than the ones proposed by Lin (see also Proposition 5.5).

## 6.1. Fractional BBM Equations

We are now ready to characterize the spectral stability of the ground state traveling wave solutions of the fractional BBM equation, which is given by

$$\partial_t u + \partial_x u + \partial_t (|\partial_x|^s u) + \partial_x (u^{p+1}) = 0 \tag{6.5}$$

<sup>11</sup>This relationship is a verification, via the Fredholm alternative, that  $(\mathcal{I} + \mathcal{M}) U_c \in \ker(\mathcal{L}_0)^\perp$

(compare with (6.1)). In the traveling wave ansatz  $U_c(\xi)$  with  $\xi = x - ct$  we obtain the existence equation

$$c|\partial_\xi|^s U_c + (c-1)U_c - U_c^{p+1} = 0. \quad (6.6)$$

Under the assumption  $c > 1$ , which is necessary for the existence of such waves, we see that matters once again reduce to the solution  $Q$  of (5.9). Indeed, the solution  $U_c$  of (6.6) can be written as

$$U_c(x) = (c-1)^{1/p} Q\left(\left[\frac{c-1}{c}\right]^{1/s} x\right).$$

Next, we verify that the operator  $\mathcal{L}_0 = c|\partial_\xi|^s + (c-1) - (p+1)U_c^p$  satisfies the requirements of Theorem 6.1 for any  $c > 1$  and  $s > 0$ . Indeed, letting the multiplier satisfy  $\alpha(\xi) = |\xi|^s$ , we see that  $c\alpha(\xi) + c - 1 \geq c - 1 =: \delta$ . Similar to the claims of Proposition 5.5, we use the  $|\xi|^{-1}$  decay of  $Q$  (and  $U_c$ , respectively) to conclude that the operator  $(I + |\partial_\xi|^s)^{-1/2} U_c^p (I + |\partial_\xi|^s)^{-1/2}$  is a compact operator on  $L^2(\mathbb{R})$ . Finally,  $\ker(\mathcal{L}_0) = \text{span}\{\partial_\xi U_c\}$  and  $n(\mathcal{L}_0) = 1$  is a consequence<sup>12</sup> of the corresponding statement for the operator  $\mathcal{L}_+$  of Frank and Lenzmann [7].

Thus, we may apply our formula for the Hamilton-Krein index from Theorem 6.1. To this end, we need to compute  $\partial_c \langle (I + \mathcal{M})U_c, U_c \rangle$ , which from the defining equation (6.6) is expressible as follows

$$\langle (I + \mathcal{M})U_c, U_c \rangle = \frac{1}{c} \langle U_c + U_c^{p+1}, U_c \rangle = (c-1)^{\frac{2}{p}-\frac{1}{s}} c^{\frac{1}{s}-1} \langle Q, Q \rangle + (c-1)^{1+\frac{2}{p}-\frac{1}{s}} c^{\frac{1}{s}-1} \langle Q^{p+1}, Q \rangle.$$

It follows that

$$\partial_c \langle (I + \mathcal{M})U_c, U_c \rangle = (c-1)^{\frac{2}{p}-\frac{1}{s}-1} c^{\frac{1}{s}-2} [(c(2/p-1) + 1 - 1/s) \langle Q, Q \rangle + (2c/p + 1 - 1/s) \langle Q^{p+1}, Q \rangle].$$

Since  $c > 1$ , upon simplifying we have that

$$\partial_c \langle (I + \mathcal{M})U_c, U_c \rangle \propto [(2-p)sc + (s-1)p] \langle Q, Q \rangle + [2sc + (s-1)p] \langle Q^{p+1}, Q \rangle.$$

Now, using the existence equation (5.9) we have that

$$\langle Q^{p+1}, Q \rangle = \langle Q, Q \rangle + \langle |\partial_\xi|^s Q, Q \rangle = \langle Q, Q \rangle + \langle |\partial_\xi|^{s/2} Q, |\partial_\xi|^{s/2} Q \rangle;$$

thus, we can rewrite the above to say that

$$\partial_c \langle (I + \mathcal{M})U_c, U_c \rangle \propto [(4-p)sc + 2(s-1)p] \langle Q, Q \rangle + [2sc + (s-1)p] \langle |\partial_\xi|^{s/2} Q, |\partial_\xi|^{s/2} Q \rangle.$$

This allows us to conclude with the following:

**Theorem 6.3.** *Let  $0 < s < 2$ ,  $p \in (0, p_{\max})$  and  $c > 1$ . The unique (up to translation) ground state  $U_c$  of the fractional BBM equation (which is a solution of (6.6)) is spectrally stable if and only if*

$$[(4-p)sc + 2(s-1)p] \langle Q, Q \rangle > -[2sc + (s-1)p] \langle |\partial_\xi|^{s/2} Q, |\partial_\xi|^{s/2} Q \rangle.$$

*In particular, the wave is spectrally stable if  $1 \leq s < 2$  and  $0 < p \leq 4$ . If the inequality is reversed the linearized eigenvalue problem has precisely one positive real eigenvalue, and the rest of the spectrum is purely imaginary.*

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<sup>12</sup>by a simple change of variables

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